A NEW CLASS OF MULTIDIMENSIONAL FLOWS OF COMPRESSIBLE MEDIA ADMITTING EXACT LINEARIZATION OF THE NAVIER–STOKES EQUATIONS

M. A. Demidov and A. P. Mikhailov

UDC 533.6

As was established in [1-4], isoentropic supercompression of a material is realized with growth in pressure on its boundary in a regime with aggravation (unlimited increase over a finite time t_f)

$$p(0,t) = p_0 (t_f - t)^{n_s}, \quad n_s = -2\gamma (N+1)/(\gamma + 1 + N(\gamma - 1)), \tag{1}$$

where p is pressure, γ is the adiabatic index, and N = 0, 1, 2 is the symmetry index. In the study of limiting regimes with aggravation in gas dynamics problems [5-10], including various physical problems [11], it has been shown that "slow" regimes with aggravation [to which Eq. (1) also applies] lead to localization of flows in a finite region, shock-free compression, and formation of gas dynamic structures [5-12], while in "rapid" regimes localization is absent and compression is accompanied by development of a shock wave, which intensifies without limit as $t \to t_f$ [13]. Study of shock-free compression and the localization effect for multidimensional gas dynamic flows is of undoubtable interest, and is the goal of the present study.

We will consider multidimensional flows of viscous compressible media with a homogeneous spatial density [$\rho = \rho_1(t)$, $\eta = 1/\rho = \eta_1(t)$, where η is the specific volume]. It has been shown that the Navier–Stokes equations then reduce to linear elliptical Poisson equations. Using the example of the one-dimensional case the characteristics of all media allowing flows with homogeneous spatial density have been analyzed. On the basis of the equations obtained solutions have been constructed which describe the effect of localization of multidimensional gas dynamic flows.

Individual questions related to study of flows with homogeneous density for the Euler equations and in the one-dimensional case were treated in greater detail in [14, 15].

The continuity equation

$$\partial\rho/\partial t + \operatorname{div}(\rho \mathbf{v}) = 0 \tag{2}$$

(where $\mathbf{v}$ is the velocity) for continuous flows with homogeneous density [$\rho = \rho_1(t) = \eta_1^{-1}(t)$] reduces to the form

$$\operatorname{div} \mathbf{v} = \frac{1}{\eta_1} \frac{d\eta_1}{dt}. \tag{3}$$

We write the general solution of Eq. (3):

$$\mathbf{v} = \mathbf{v}_0 + \operatorname{rot} \mathbf{A}(\mathbf{r}, t), \quad (v_0)_i = \alpha_i^k(t)\, r_k, \quad \operatorname{Tr}\|\alpha_i^k\| = \frac{1}{\eta_1}\frac{d\eta_1}{dt}. \tag{4}$$

Here N + 1 is the dimensionality of the space; $\{r_1, \ldots, r_N\}$ are Euler coordinates; $\mathbf{A}(\mathbf{r}, t)$ is an arbitrary vector function.

Assuming that the dynamic viscosity of the gas depends only on its density $\mu = \mu(\rho_1(t))$, we make use of the Navier–Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} + [\operatorname{rot}\mathbf{v}, \mathbf{v}] + \operatorname{grad}\frac{\mathbf{v}^2}{2} = -\frac{1}{\rho}\operatorname{grad} p + \left(\mu + \frac{2}{3}\xi\right)\operatorname{grad}\operatorname{div}\mathbf{v} + \mu\Delta\mathbf{v} - \operatorname{grad} U \tag{5}$$

[where $\xi = \xi(\rho)$ is the second viscosity coefficient, arbitrary in the general case, and U is the potential of external forces].

In view of the assumption $\rho = \rho_1(t)$ and the satisfaction of Eq. (3) $\operatorname{grad}\operatorname{div}\mathbf{v} \equiv 0$ and $\rho^{-1}\operatorname{grad} p = \operatorname{grad}(p\rho^{-1})$, consequently Eq. (5) can be reduced to the form

---

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 70-75, May-June, 1988. Original article submitted March 27, 1987.

0021-8944/88/2903-0378$12.50

$$\frac{\partial \mathbf{v}}{\partial t} + [\mathrm{rot}\, \mathbf{v}, \mathbf{v}] - \mu(\rho)\, \Delta \mathbf{v} = -\mathrm{grad}\, \Psi,\ \Psi = \frac{p}{\rho_1} + \frac{\mathbf{v}^2}{2} + U. \tag{6}$$

The condition of solvability of rot grad $\Psi \equiv 0$ yields an equation for the velocity

$$\frac{\partial}{\partial t}\, \mathrm{rot}\, \mathbf{v} + \mathrm{rot}\, [\mathrm{rot}\, \mathbf{v}, \mathbf{v}] - \mu(\rho)\, \mathrm{rot}\, \Delta \mathbf{v} = 0, \tag{7}$$

which according to Eq. (4) in fact defines the vector function $\mathbf{A}(\mathbf{r}, t)$. The potential $\Psi(\mathbf{r}, t)$, obtained to the accuracy of an insignificant scalar function of time $F_1(t)$, can be found from the linear elliptical Poisson equation

$$\Delta \Psi = -\frac{d}{dt}\left(\frac{1}{\eta_1}\frac{d\eta_1}{dt}\right) - \mathrm{div}\, [\mathrm{rot}\, \mathbf{v}, \mathbf{v}], \tag{8}$$

where the specific volume η_1 and velocity $\mathbf{v}$ appear on the right side as parameters, which is also valid for multidimensional flows of ideal compressible media (described by the Euler equations), since consideration of viscosity forces does not affect its form.

The analysis simplifies significantly in the case of potential flows

$$\mathbf{v} = \mathrm{grad}\, \Phi, \tag{9}$$

for which Eq. (7) is satisfied identically, while Eq. (9) also leads to a linear Poisson equation

$$\Delta \Phi = \frac{1}{\eta_1}\frac{d\eta_1}{dt} \tag{10}$$

and Eq. (8) becomes

$$\Delta \Psi = -\frac{d}{dt}\left(\frac{1}{\eta_1}\frac{d\eta_1}{dt}\right). \tag{11}$$

Thus for the flows studied integration of the Navier–Stokes equations reduces to solution of the classical linear equations (10), (11).

Clarification of the conditions required for realization of flows with homogeneous spatial density for various models of continuous media is of special interest. The method for determining the characteristics of all media admitting flows with homogeneous density consists of the following.

1. Assuming that the gas density is homogeneous [$\rho = \rho_1(t)$] general explicit solutions of the equations of motion and continuity are constructed, depending parametrically on the specific volume function $\eta_1(t)$ [in the multidimensional case the solutions of the corresponding problems for Eqs. (10), (11)].

2. For arbitrary energy balance equations in the medium the explicit solutions permit establishing the general functional form of the characteristics of all media [equations of state $\varepsilon(p, \eta)$, $T(p, \eta)$, thermal conductivity coefficients $\varkappa(p, \eta)$, thermal flux relaxation $\tau(p, \eta)$, sources $Q(p, \eta)$, etc.] admitting flows with homogeneous density.

3. For the medium found in this manner the energy equation reduces to an ordinary differential equation which defines the time behavior of the function $\eta_1(t)$, and consequently, the complete form of the solutions obtained. We will present the major results. For adiabatic flows of ideal media the equations of state

$$\varepsilon(p, \eta) = p\varepsilon_1(\eta) + \varepsilon_2(\eta) \tag{12}$$

[where $\varepsilon_1(\eta)$ and $\varepsilon_2(\eta)$ are arbitrary functions] are admissible. In particular, for $\varepsilon_1(\eta) = \eta/(\gamma - 1)$, $\varepsilon_2(\eta) \equiv 0$, Eq. (12) describes an ideal perfect gas, and for $\varepsilon_1(\eta) = \eta/(\gamma - 1)$, $\varepsilon_2(\eta) = a\eta/(\eta - b)^2$, a Van der Waals gas. The function $\eta_1(t)$ is defined by the quadrature

$$t = \pm \frac{1}{N+1}\int \eta_1^{\frac{N}{N+1}}\left(\exp\left(-\int \frac{d\eta_1}{\varepsilon_1(\eta_1)}\right) + C\right)^{-1/2} d\eta_1, \quad C = \mathrm{const}. \tag{13}$$

For a perfect thermally conductive gas [$p = \rho RT$, $\varepsilon = p\eta/(\gamma - 1)$] the thermal conductivity coefficient

$$\varkappa(\eta, T) = T\varkappa_1(\eta) \tag{14}$$

[where $\varkappa_1(\eta)$ is an arbitrary function]. The specific volume $\eta_1(t)$ is found from a nonlinear autonomous third-order equation

$$\varkappa_1(\eta_1) = -\frac{1}{N+1}\frac{R^2}{\gamma-1}\eta_1^{-\gamma}\left(\left(\eta_1^{\frac{1}{N+1}}\right)''\right)^{-2}\left(\eta_1^{\gamma-\frac{N}{N+1}}\left(\eta_1^{\frac{1}{N+1}}\right)''\right)'. \tag{15}$$

In the general case (thermally conductive gas with sources) for characteristics of the form

$$\varepsilon(p,\eta) = p\varepsilon_1(\eta),\ T(p,\eta) = pT_1(\eta),$$
$$\varkappa(p,\eta) = p\varkappa_1(\eta),\ Q(p,\eta) = pQ_1(\eta), \tag{16}$$

where $\varepsilon_1(\eta)$, $T_1(\eta)$, $\varkappa_1(\eta)$, $Q_1(\eta)$ are arbitrary functions, and we obtain $\eta_1(t)$ from the equation

$$\eta_1''\varepsilon_1(\eta_1) = -(\eta_1')^2/2 - \int\{(\eta_1'')^2\eta_1^{-1}T_1(\eta_1)\,3\varkappa_1(\eta_1) - \eta_1\eta_1''Q_1(\eta_1)\}\,dt. \tag{17}$$

We will note that consideration of the thermal flux relaxation effect (within the framework of hyperbolic heat transport) is possible for coefficients $\tau(p,\eta) = \tau_1(\eta)$, $\varkappa(p,\eta) = p\varkappa_1(\eta)$ [($\tau_1(\eta)$, $\varkappa_1(\eta)$ are arbitrary functions].

Solutions can also be constructed for media with characteristics more general than those of Eq. (16) (for arbitrary functions $\varepsilon(p,\eta)$, $\varkappa(p,\eta)$, $T(p,\eta)$, [14]). Equations (13), (15), (17) have particular solutions of a power form

$$\eta_1(t) = \eta_0(t_f - t)^\alpha, \tag{18}$$

which for $0 < \alpha < 1$ describe regimes with aggravation [for adiabatic flows of an ideal gas $\alpha = 2/(\gamma + 1)$].

We will now consider the concrete formulation of boundary problems for Eqs. (10), (11). We will seek solutions in which the pressure and velocity of the material vanish simultaneous on some closed boundary $\partial\Omega_2$ (in view of the condition $|\mathbf{v}|\,|_{\partial\Omega_2} \equiv 0$ the boundary $\partial\Omega_2$ is immobile). The problem consists of defining velocity and pressure fields in the region external to $\partial\Omega_2$ and finding the pressure and velocity distributions on the fixed boundary $\partial\Omega_1(t)$, corresponding, for example, to an external piston compressing the gas.

The boundary problems formulated for Eqs. (10), (11) can be written in the form

$$\Delta\Psi = -\frac{d}{dt}\left(\frac{1}{\eta_1}\frac{d\eta_1}{dt}\right),\quad \Psi|_{\partial\Omega_2} = 0,\quad \Delta\Phi = \frac{1}{\eta_1}\frac{d\eta_1}{dt},\quad \left.\frac{\partial\Phi}{\partial n_2}\right|_{\partial\Omega_2} = 0 \tag{19}$$

(where n_2 is the normal to $\partial\Omega_2$). A solution of the classical problems of Eq. (19) exists and is unique. The pressure distribution on the piston surface is found from

$$p(\mathbf{r},t) = \{\Psi - U(\mathbf{r},t) - (\mathrm{grad}\,\Phi)^2/2\}\eta_1(t). \tag{20}$$

In view of the time-independence of the boundary conditions the solutions of Eq. (19) are constructed by separating the variables $\mathbf{r}$ and $\eta_1(t)$:

$$\Psi = -\lambda_2^{-1}\frac{d}{dt}\left(\frac{1}{\eta_1}\frac{d\eta_1}{dt}\right)\Psi_2(\mathbf{r}),\quad \Phi = \lambda_1^{-1}\frac{1}{\eta_1}\frac{d\eta_1}{dt}\Phi_2(\mathbf{r}); \tag{21}$$

$$\Delta\Phi_2(\mathbf{r}) = \lambda_1,\quad \left.\frac{\partial\Phi_2}{\partial n_2}\right|_{\partial\Omega_2} = 0,\quad \Delta\Psi_2(\mathbf{r}) = \lambda_2,\quad \Psi_2(\mathbf{r})|_{\partial\Omega_2} = 0. \tag{22}$$

In the case of a power function for η_1, Eq. (18), the pressure and velocity are calculated with with the expressions

$$\mathbf{v} = \alpha\lambda_1^{-1}(t_f - t)^{-1}\,\mathrm{grad}\,\Phi_2(\mathbf{r}),\quad \eta_1(t) = \eta_0(t_f - t)^\alpha,$$
$$p = \eta_0^{-1}\alpha\lambda_2^{-1}(t_f - t)^{-\alpha-2}\{\Psi_2(\mathbf{r}) - (\mathrm{grad}\,\Phi_2(\mathbf{r}))^2/2\}. \tag{23}$$

The solutions of Eq. (23) are an example of localization of multidimensional gas dynamic processes for compression of a medium in a regime with aggravation. The velocity, density, and pressure of the gas increase without limit upon approach to the final time t_f for $\alpha > 0$. In view of the boundary conditions of Eq. (19) the gas dynamic motion is localized in a region between the piston $\partial\Omega_1(t)$ and the front $\partial\Omega_2$ (for more detailed information on the localization effect in gas dynamics see [5-12]).

Further analysis involves distinguishing radially symmetric solutions of system (22), dependent solely on $r = |\mathbf{r}|$:

$$\Phi_2^1(r) = \begin{cases} \frac{\lambda_1}{N+1}\frac{r^2}{2} + \frac{C_1}{1-N}r^{1-N} + C_2, & N \neq 1, \\ \frac{\lambda_1}{N+1}\frac{r^2}{2} + C_3 \ln r + C_4, & N = 1, \end{cases}$$
$$\Psi_2^1(r) = \begin{cases} \frac{\lambda_2}{N+1}\frac{r^2}{2}\frac{C_5}{1-N}r^{1-N} + C_6, & N \neq 1, \\ \frac{\lambda_2}{N+1}\frac{r^2}{2} + C_7 \ln r + C_8, & N = 1. \end{cases} \quad (24)$$

If the boundary $\partial\Omega_1$ is a sphere with radius r_0, then Eq. (24) (for corresponding choice of the constant C_i) gives a complete solution of Eq. (19):

$$\mathbf{v}(\mathbf{r}, t) = -\frac{\alpha}{N+1}\frac{1}{t_f - t}\mathbf{r}(1 - (r_0/r)^{N+1}),$$
$$p(\mathbf{r}, t) = \eta_0^{-1}(t_f - t)^{-\alpha-2}\left(\frac{\alpha}{N+1}\frac{r^2 - r_0^2}{2} - \frac{r^2}{2}\left(1 - \left(\frac{r_0}{r}\right)^{N+1}\right)^2 - \frac{\alpha}{N+1}\right)\begin{cases} \frac{r_0^{N+1}}{1-N}(r^{1-N} - r_0^{1-N}), & N \neq 1, \\ \ln(r/r_0), & N = 1. \end{cases} \quad (25)$$

The presence of the comparison theorem for the linear problems of Eq. (22) and the precise solutions of Eqs. (24), (25) permits construction of a wide class of estimates for Eq. (19).

We will note that aside from the solution of the problem of a piston [15] Eq. (25) and in the more general case the solution of Eq. (19) describe the process of limitless concentration of matter and energy in a closed (finite) region of space. In fact, let $R_0 > r_0$, the radius of the immobile boundary $\partial\Omega_1$, on which the velocity is specified,

$$|\mathbf{v}(R_0, t)| = -\frac{\alpha}{N+1}\frac{1}{t_f - t}R_0\left(1 - \left(\frac{r_0}{R_0}\right)^{N+1}\right) \to \infty, \quad t \to t_f.$$

Then in the space between $R_0(\partial\Omega_1)$ and $r_0(\partial\Omega_2)$ over a finite time t_f the density, pressure, and velocity of the material increase without limit (in the regime with aggravation). It follows from Eq. (25) that the velocity of the closed boundary attached to fixed particles of the material (the piston) changes by a law

$$v_p(t) = -\frac{\alpha}{N+1}(t_f - t)^{\alpha-1}C_1(r_0^{N+1} + C_1(t_f - t)^{\alpha})^{-N/(N+1)} \quad (27)$$

($C_1 > 0$ is a constant defined by the mass of the compressed gas or initial position of the piston).

At the beginning of the compression process the piston velocity and, as can be simply shown, the pressure change by a law corresponding to one-dimensional solutions in separable mass and time variables (1) ([1-10], N = 0, 1, 2). For $r_p(t) \to r_0$ [where $r_p(t)$ is the piston coordinate] the pressure on the piston tends to the law for planosymmetric flows (1) (N = 0), which have the property of localization [5, 6, 9, 10].

In conclusion, we will note that for flows with homogeneous density [in view of the equality $\operatorname{div} \mathbf{v} = (\ln \eta_1)' = f_1(t))$] consideration of the contribution of viscous forces to the energy balance equation in the medium simplified considerably. The results of the present study indicate that in viscous compressible media, flows with homogeneous density are described by classical linear elliptic equations and that the localization and shock-free supercompression effects are realized.

The authors are indebted to S. P. Kurdyumov and N. V. Zmitrenko for their valuable advice.

LITERATURE CITED

1. N. V. Zmitrenko and S. P. Kurdyumov, "Self-similar regimes of plasma compression by a piston," Heat and Mass Transport [in Russian], Vol. 8, ITMO Akad. Nauk BSSR, Minsk (1972).
2. S. K. Zhdanov and B. A. Trubnikov, "Optimal plasma compression in Z- and θ -pinches," Pis'ma Zh. Éksp. Teor. Fiz., 21, No. 6 (1975).
3. N. V. Zmitrenko and S. P. Kurdyumov, "N- and S-regimes of self-similar compression of a finite plasma mass and the singularity of regimes with aggravation," Zh. Prikl. Mekh. Tekh. Fiz., No. 1 (1977).

4. Ya. M. Kazhdan, "On the question of adiabatic gas compression under the action of a spherical piston," Zh. Prikl. Mekh. Tekh. Fiz., No. 1 (1977).
5. M. A. Anufrieva, M. A. Demidov, A. P. Mikhailov, and V. V. Stepanova, Regimes with Aggravation in Gas Dynamics Problems. Mathematical Models, Analytic and Numerical Methods in Transport Theory [in Russian], ITMO Akad. Nauk BSSR, Minsk (1982).
6. M. A. Demidov, Yu. A. Klokov, and A. P. Mikhailov, "Shock-free compression of a finite gas mass by a planar piston with arbitrary entropy distribution," Preprint Inst. Prikl. Mat. Akad. Nauk SSSR [in Russian], No. 151, Moscow (1984).
7. M. A. Demidov, Yu. A. Klokov, and A. P. Mikhailov, "Structures under shock-free spherical compression of a gas with arbitrary entropy distribution," Preprint Inst. Prikl. Mat. Akad. Nauk SSSR [in Russian], No. 73, Moscow (1985).
8. M. A. Demidov and A. P. Mikhailov, "An exact solution describing complex gas dynamic structures," Inzh.-Fiz. Zh., 51, No. 6 (1986).
9. M. A. Demidov, A. P. Mikhailov, and V. V. Stepanova, "Localization and structure for gas compression in a regime with aggravation," Dokl. Akad. Nauk SSSR, 281, No. 1 (1985).
10. M. A. Demidov and A. P. Mikhailov, "Localization and structure formation effects in adiabatic compression of a finite gas mass in a regime with aggravation," Prikl. Mat. Mekh., 50, No. 1 (1986).
11. M. A. Demidov, "Construction of solutions describing the localization effect in some compressible media," Preprint Inst. Prikl. Mat. Akad. Nauk SSSR [in Russian], No. 1, Moscow (1986).
12. A. P. Mikhailov and V. V. Stepanova, "Localization of gas dynamic processes and structures in gas compression in a regime with aggravation," Prikl. Mat. Mekh., 48, No. 6, (1984).
13. A. P. Mikhailov and V. V. Stepanova, "On a self-similar gas dynamics problem," Preprint Inst. Prikl. Mat. Akad. Nauk SSSR [in Russian], No. 162, Moscow (1985).
14. M. A. Demidov, "Flow of a homogeneous density gas through space," Preprint Inst. Prikl. Mat. Akad. Nauk SSSR [in Russian], No. 126, Moscow (1985).
15. M. A. Demidov and A. P. Mikhailov, "Multidimensional flows with homogeneous density and the localization effect," Inst. Prikl. Mat. Akad. Nauk SSSR [in Russian], No. 53, Moscow (1986).

EXPERIMENTAL STUDY OF SUPERSONIC THREE-DIMENSIONAL JETS

G. M. Zhinzhikov and N. O. Pavlova

UDC 533.17

The interest in the study of three-dimensional jets, i.e., jets in which the three-dimensional character of flow is due to the form of the outlet section of the nozzle [1], stems from their increasing practical value. For example, such nozzles are used in modern supersonic aircraft [2], in the gas-processing industry [3], and in other applications.

There have been relatively few experimental studies of the propagation of three-dimensional jets; of the studies that have been conducted, we can note [4-6], with the latter being the most complete.

Here, we experimentally study the shock-wave structure and parameter distribution in supersonic underexpanded jets of cold air (T_0 ~ 290 K) discharged into the atmosphere (p_∞ ~ 0.1 MPa) from rectangular sonic nozzles. We used schlieren visualization of the flow and we measured the total head on the jet axis. Empirical relations were obtained to determine the position of the central discontinuity in three-dimensional jets and the Mach-number distribution on the axis. The results are compared with the data in [6].

In our experiments, we used sonic nozzles with a rectangular edge and a ratio of sides of the rectangle λ equal to 1, 2, 3, 5, and 10. This ratio is referred to below as the elongation of the nozzle edge. The size of the lesser side was 6-12 mm. The nozzle took the form of a rectangular opening in the end of a cylinder with an inside diameter of 80 mm. The nozzle had shaped subsonic and equalizing plane-parallel sections about 4 mm long. The

Leningrad. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 75-80, May-June, 1988. Original article submitted January 30, 1987.

 0021-8944/88/2903-0382$12.50